Sufficient Conditions for Solving Linearly Constrained Separable Concave Global Minimization Problems

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Abstract. A concave function defined on a polytope may have many local minima (in fact every extreme point may be a local minimum). Sufficient conditions are given such that if they are satisfied at a point, this point is known to be a global minimum. It is only required to solve a single linear program to test whether the sufficient conditions are satisfied. This test has been incorporated into an earlier algorithm to give improved performance. Computational results presented show that these sufficient conditions are satisfied for certain types of problems and may substantially reduce the effort needed to find and recognize a global minimum.

Key words. Global minimization, sufficient conditions.

1. Introduction

This paper considers the problem of finding sufficient conditions for obtaining a solution to a linearly constrained concave global minimization problem of the form

$$\operatorname*{global\,min}_{(x,y)\in\Omega} \psi(x, y) = \varphi(x) + d^t y\,, \tag{GP}$$

where the nonlinear term $\varphi(x)$ can be expressed in separable form. That is,

$$\varphi(x) = \sum_{i=1}^{n} \varphi_i(x_i)$$

and each $\varphi_i(x_i)$ is concave. Additionally, $\Omega = \{(x, y): A_1 x + A_2 y \leqslant b,\ x \geqslant 0,\ y \geqslant 0\}$ is assumed to be nonempty and bounded, and $x \in \mathbf{R}^n$, $y \in \mathbf{R}^p$, $A_1 \in \mathbf{R}^{m\times n}$, $A_2 \in \mathbf{R}^{m\times p}$, $b \in \mathbf{R}^m$, and $d \in \mathbf{R}^p$.

Problem (GP) is a constrained combinatorial optimization problem for which many well known problems are special cases. For example, the concave quadratic global minimization problem is a special case of problem (GP) for which $\varphi(x) = (1/2)x^t Q x + c^t x$ where $Q \in \mathbf{R}^{n\times n}$ is symmetric and negative definite ($\varphi(x)$ can be transformed into separable forms using the eigenstructure of Q). This concave quadratic global minimization problem is known to be NP-hard (Phillips 1988), and hence it follows that problem (GP) is NP-hard. From a computational viewpoint, this means that, in the worst case, the computing time required to

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obtain a solution will grow exponentially with the number of nonlinear variables. An important property of problem (GP), which is basic to many solution methods (Phillips 1988), is that the global minimum point is always found at a vertex of the convex polytope Ω. For this reason, linear programming is an essential part of any computational algorithm to solve problem (GP).

Additionally, many practical problems in engineering design can be formulated as problem (GP). For example, a class of synomial optimization problems, for which the theory of geometric programming was originally developed (Duffin, Peterson, and Zener 1967), can be transformed into problems of the form (GP). Such problems occur in engineering design, where the function to be minimized can often be expressed as the sun of component costs of the form

$$u_i = c_i t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_m^{a_{im}},$$

where the constants a_{ij} are specified (possibly negative) real constants, and the design parameters t_i are assumed to be positive variables. If the constants $c_i \leq 0$ for all i, then this function in turn can easily be converted, using standard techniques (Phillips and Rosen 1990), to a concave function in the form required by problem (GP).

In this paper sufficient conditions for recognizing a solution to the global minimization problem (GP) are presented and justified. It is shown that the use of these sufficient conditions to recognize a global minimum can significantly accelerate the solution for certain types of problems of the form (GP). This is supported by computational results for problems where the linear terms tend to dominate the nonlinear terms in the objective function. Thus it is shown that these sufficient conditions are satisfied for certain types of problems, and that they can be used to substantially improve the performance of an earlier algorithm. Hence, a new algorithm is presented which incorporates a termination test based on satisfaction of the sufficient conditions.

The sufficient conditions can be best understood in terms of solving a multiple cost row linear program with $O(2^n)$ cost rows. However, it will be shown that it is only necessary to solve a single linear program to determine if the sufficient conditions are satisfied. If they are not satisfied, this information can frequently be used to obtain improved bounds and possibly eliminate part of the feasible set from further consideration.

In the next section, the sufficient conditions are presented and justified. In Section 3, a detailed description of the new algorithm is presented. In Section 4, error bounds are used to show that an ε-approximate solution is obtained after a finite number of steps. A simple numerical example is given in Section 5. Finally, in Section 6 the computational results are presented. These results are summarized in two tables. In Table I results are given for the most difficult type of concave quadratic problem; that is, where the quadratic term dominates and has its maximum at an interior feasible point. It is seen that the new algorithm only gives a faster solution in a few cases. It gives much faster solution times when the

linear term dominates, as shown in Table II. Comparisons were made for five problems and the new algorithm was faster in all cases. In two of the give problems the time was reduced by a factor of three.

2. Sufficient Conditions

Let $(x^*, y^*) \in \Omega$ be a global optimum solution, with corresponding function value $\psi^* = \psi(x^*, y^*)$, to the problem (GP). Let $R \subset \mathbf{R}^n$ be a hyperrectangle defined by

$$R = \{x : \beta_{i_1} \leq x_i \leq \beta_{1_2}, \quad i = 1, \ldots, n\},$$

where the lower and upper bounds β_{i_1} and β_{i_2} are given (see Section 3 for details on how to easily obtain these bounds). Also, define the linear underestimator of $\varphi(x)$ over R to be

$$\Gamma(x) = \sum_{i=1}^{n} \gamma_i(x_i)$$

where each $\gamma_i(x_i)$ is a linear function such that $\gamma_i(x_i) \leq \varphi_i(x_i)$ for all $x_i \in [\beta_{i_1}, \beta_{i_2}]$, $\gamma_i(\beta_{i_1}) = \varphi_i(\beta_{i_1})$, and $\gamma_i(\beta_{i_2}) = \varphi_i(\beta_{i_2})$. Hence, $\Gamma(x)$ is a linear underestimator of $\varphi(x)$ over R which agrees with $\varphi(x)$ at all vertices of R. Denote $\gamma_i(x_i) = c_{i_0} x_i + \gamma_{i0}$ for $i = 1, \ldots, n$, so that $\gamma_{i_0} = (\beta_{i_2}\varphi_i(\beta_{i_1}) - \beta_{i_1}\varphi_i(\beta_{i_2}))/(\beta_{i_2} - \beta_{i_1})$ and $c_{i_0} = (\varphi_i(\beta_{i_2}) - \varphi_i(\beta_{i_1}))/(\beta_{i_2} - \beta_{i_1})$.Note that $\gamma_i(x_i)$ is the convex envelope of $\varphi_i(x_i)$ over $[\beta_{i_1}, \beta_{i_2}]$, and hence $\Gamma(x)$ is the convex envelope of $\varphi(x)$ over R. Now let $R' = R \times \mathbf{R}^p$, and solve the linear program

$$\Gamma(x') + d^t y' = \min_{(x,y) \in \Omega \cap \mathrm{R}'} \Gamma(x) + d^t y .$$

Clearly if $(x^*, y^*) \in \Omega \cap \mathrm{R}'$, then $\Gamma(x') + d^t y' \leq \psi^* \leq \psi(x', y')$. If it also happens that $\psi(x', y') - (\Gamma(x') + d^t y') \leq \varepsilon$ for some small user specified $\varepsilon \geq 0$, then (x', y') is usually accepted as the global optimum solution. In many of the recently proposed computational methods (Phillips and Rosen 1988, 1990), the solution (x', y') obtained from this linear underestimating problem has, in fact, turned out to be the global optimum solution to problem (GP). Unfortunately, the difference $\psi(x', y') - (\Gamma(x') + d^t y')$usually exceeds ε for many iterations. These iterations are required only to verify, by improving the lower bound, that the point (x', y') is indeed the global optimum solution. Clearly, it would be desirable to obtain some sufficient conditions under which the point (x', y') would be guaranteed to be the global optimum solution.

Let $\gamma_i^{(1)}(x_i)$ be the line passing through $[\beta_{i_1}, \varphi_i(\beta_{i_1})]$ and $[x_i', \varphi_i(x_i')]$, and let $\gamma_i^{(2)}(x_i)$ be the line passing through $x_i', \varphi_i(x_i')]$ and $[\beta_{i_2}, \varphi_i(\beta_{i_2})]$. Then the two linear functions $\gamma_i^{(1)}(x_i)$ and $\gamma_i^{(2)}(x_i)$ can be expressed as $\gamma_i^{(1)}(x_i) = c_{i_1} x_i + \gamma_{i_1}$ and $\gamma_i^{(2)}(x_i) = c_{i_2} x_i + \gamma_{i_2}$, where $\gamma_{i_1} = (x_i'\varphi_i(\beta_{i_1}) - \beta_{i_1}\varphi_i(x_i'))/(x_i' - \beta_{i_1})$, $\gamma_{i_2} = (\beta_{i_2}\varphi_i(x_i') - x_i'\varphi_i(\beta_{i_2}))/(\beta_{i_2} - x_i')$, and

$$c_{i_1} = \begin{cases} c_{i_0} & \text{if } x_i' \in \{\beta_{i_1}, \beta_{i_2}\} \\ (\varphi_i(x_i') - \varphi_i(\beta_{i_1}))/(x_i' - \beta_{i_1}) & \text{if } x_i' \in (\beta_{i_1}, \beta_{i_2}) \end{cases} \quad \text{and}$$

$$c_{i_2} = \begin{cases} c_{i_0} & \text{if } x_i' \in \{\beta_{i1}, \beta_{i_2}\} \\ (\varphi_i(x_i') - \varphi_i(\beta_{i_2}))/(x_i' - \beta_{i_2}) & \text{if } x_i' \in (\beta_{i_1}, \beta_{i_2}) \end{cases}$$

Hence, $\gamma_i^{(1)}(x_i)$ is a linear underestimator of $\varphi_i(x_i)$ over $[\beta_{i_1}, x_i']$ and agrees with $\varphi_i(x_i)$ at β_{i_1} and x_i'. Likewise, $\gamma_i^{(2)}(x_i)$ is a linear underestimator of $\varphi_i(x_i)$ over $[x_i', \beta_{i_2}]$ and agrees with $\varphi_i(x_i)$ at x_i' and β_{i_2}. Notice that if either $x_i' = \beta_{i_1}$ or $x_i' = \beta_{i_2}$, then $\gamma_i(x_i) \equiv \gamma_i^{(1)}(x_i) \equiv \gamma_i^{(2)}(x_i)$. Assume, without loss of generality, that $x_i' \in (\beta_{i_1}, \beta_{i_2})$ for $i = 1, \ldots, n_0$ and that $x_i' \in \{\beta_{i_1}, \beta_{i_2}\}$ for $i = n_0 + 1, \ldots, n$. For completeness, also note that $c_{i_1} \geqslant c_{i_0} \geqslant c_{i_2}$ for $i = 1, \ldots, n_0$.

THEOREM 1. *Consider the multiple-cost-row linear program with 2^{n_0} distinct cost rows*

$$\min_{(x,y)\in\Omega} \sum_{i=1}^{n_0} c_{ij_i} x_i + \sum_{i=n_0+1}^{n} c_{i_0} x_i + d^t y\,, \tag{MCRLP}$$

where $j_i \in \{1, 2\}$ for $i = 1, \ldots, n_0$. If $x^* \in R$ and if (x', y') solves (MCRLP) for all 2^{n_0} distinct cost rows, then (x', y') satisfies $\psi^* = \psi(x', y')$; that is, (x', y') is a global minimum vertex for problem (GP).

Proof. Since $\varphi_i(x_i)$, $\gamma_i^{(1)}(x_i)$, and $\gamma_i^{(2)}(x_i)$ agree at x_i' for $i = 1, \ldots, n$, then

$$\begin{aligned} \psi(x', y') &= \varphi(x') + d^t y' \\ &= \sum_{i=1}^{n_0} c_{ij_i} x_1' + \sum_{i=n_0+1}^{n} c_{i_0} x_i' + d^t y' + \sum_{i=1}^{n_0} \gamma_{ij_i} + \sum_{i=n_0+1}^{n} \gamma_{i_0} \end{aligned}$$

for all $j_i \in \{1, 2\}$, $i = 1, \ldots, n_0$. Furthermore, if $(x', y') \in \Omega$ is the minimum of (MCRLP), and since $(x^*, y^*) \in \Omega$, we get that

$$\begin{aligned} \psi(x', y') &\leqslant \sum_{i=1}^{n_0} c_{ij_i} x_i^* + \sum_{i=n_0+1}^{n} c_{i_0} x_i^* + d^t y^* + \sum_{i=1}^{n_0} \gamma_{ij_i} + \sum_{i=n_0+1}^{n} \gamma_{i_0} \\ &= \sum_{i=1}^{n_0} \gamma_i^{(j_i)}(x_i^*) + \sum_{i=n_0+1}^{n} \gamma_i(x_i^*) + d^t y^* \end{aligned}$$

for all $j_i \in \{1, 2\}$, $i = 1, \ldots, n_0$. But $\gamma_i^{(1)}(x_i) \leqslant \varphi_i(x_i)$ over $[\beta_{i_1}, x_1']$ and $\gamma_i^{(2)}(x_i) \leqslant \varphi_i(x_i)$ over $[x_i', \beta_{1_2}]$, and since $x_i^* \in [\beta_{i_1}, \beta_{i_2}]$ for all $i = 1, \ldots, n$, then there exists a set $\{j_1, j_2, \ldots, j_{n_0}\}$ (where $j_i \in \{1, 2\}$ for $i = 1, \ldots, n_0$) such that $\gamma_i^{(j_i)}(x_i^*) \leqslant \varphi_i(x_i^*)$ for $i = 1, \ldots, n_0$. In addition, it is clear that $\gamma_i(x_i^*) \leqslant \varphi_i(x_i^*)$ for $i = n_0 + 1, \ldots, n$. Hence, there is some set $\{j_1, j_2, \ldots, j_{n_0}\}$, where $j_i \in \{1, 2\}$ for $i = 1, \ldots, n_0$, such that

$$\sum_{i=1}^{n_0} \gamma_i^{(j_i)}(x_i^*) + \sum_{i=n_0+1}^{n} \gamma_i(x_i^*) + d^t y^* \leqslant \sum_{i=1}^{n} \varphi_i(x_i^*) + d^t y^* = \psi(x^*, y^*)\,.$$

Thus, $\psi(x', y') \leqslant \psi^* \leqslant \psi(x', y')$ and so $\psi^* = \psi(x', y')$. □

Checking the optimality of (x', y') for problem (MCRLP) for all 2^{n_0} cost rows obviously requires an exponential amount of computation. Theorem 2 below presents an alternate optimality check that can be performed in polynomial time. It uses the fact that a basis remains optimal when the cost coefficients are changed, provided the reduced costs remain non-negative. For the purposes of this theorem, let B be the ordered set of indices of the basic variables at the point (x', y'), let B_i represent the index of the ith basic variable in B, and let $A_B \in \mathbf{R}^{m \times m}$ be the matrix of columns from $A = [A_1 : A_2]$ which correspond to the basic variables in B. Also recall that $c_{i_1} = c_{i_2} = c_{i_0}$ for $i = n_0 + 1, \ldots, n$, and for notational simplicity and without loss of generality, let $c_{i_1} = c_{i_2} = c_{i_0} = d_i$ for $i = n + 1, \ldots, n + p$ (in fact, all other variables x_i which are not original problem variables, e.g., slack and surplus variables if the Simplex Method is used, must have $c_{i_1} = c_{i_2} = c_{i_0} = 0$ as well). Define $w^{(j)} = A_B^{-1} a^{(j)}$ where $a^{(j)}$ is the jth column of A, and for each $j \notin B$, let $z^{(j)}$ be such that

$$z_i^{(j)} = \begin{cases} c_{i_1} & \text{if } i = B_k \text{ and } w_k^{(j)} > 0 \\ c_{i_2} & \text{otherwise.} \end{cases}$$

THEOREM 2. *For all j, define the function $g_B^{(j)}$ such that*

$$g_B^{(j)}(z) = z_j - \sum_{i=1}^{m} w_i^{(j)} z_{B_i},$$

If $x^* \in R$ and $g_B^{(j)}(z^{(j)}) \geq 0$ for all $j \notin B$, then (x', y') is a global minimum vertex for problem (GP).

Proof. First note that $g_B^{(j)}(z^{(j)}) \equiv 0$ for all $J \in B$. For each $j \notin B$, the point $z^{(j)}$ solves the linear program

$$\min_{z \in C} g_B^{(j)}(z),$$

where $C = \{z : c_{i_2} \leq z_i \leq c_{i_1},\ i = 1, 2, \ldots\}$. Hence, if $g_B^{(j)}(z^{(j)}) \geq 0$ then

$$z_j \geq \sum_{i=1}^{m} w_i^{(j)} z_{B_i} = \sum_{i=1}^{m} (A_B^{-1} a^{(j)})_i z_{B_i}$$

for any set of $z_i \in \{c_{i_1}, c_{i_2}\}$ for $i = 1, 2, \ldots,$ and $j \notin B$. Equivalently, for any $j \notin B$, the "reduced cost" r_j satisfies

$$r_j \equiv z_j - \sum_{i=1}^{m} (A_B^{-1} a^{(j)})_i z_{B_i} \geq 0$$

for any set of $z_i \in \{c_{i_1}, c_{i_2}\}$ for $i = 1, 2, \ldots$. Hence, (x', y') is optimal for problem (MCRLP) for all 2^{n_0} cost rows, and so by Theorem 1, is also optimal for problem (GP). □

If the conditions of Theorem 2 are not met, then (x', y') may not be the global optimum vertex for problem (GP). In that case, a method for obtaining a better

solution than (x', y') is needed. The method presented here for improving the solution is based in part on the work from Phillips and Rosen (1988). Specifically, let $N = \{r : r \notin B \text{ and } g_B^{(r)}(z^{(r)}) < 0\}$. For each $i \in N \cup B$, let the points $(x^{(1,i)}, y^{(1,i)})$ and $(x^{(2,i)}, y^{(2,i)})$ denote the solutions to the pair of linear programs

$$\min_{(x,y)\in\Omega\cap R'} \sum_{\substack{j=1\\ j\neq i}}^{n} \gamma_j(x_j) + \gamma_i^{(1)}(x_i) + d^t y \quad \text{and}$$

$$\min_{(x,y)\in\Omega\cap R'} \sum_{\substack{j=1\\ j\neq i}}^{n} \gamma_j(x_j) + \gamma_i^{(2)}(x_i) + d^t y$$

respectively. By defining

$$\Gamma^{(1,i)} = \sum_{\substack{j=1\\ j\neq i}}^{n} \gamma_j(x_j^{1,i}) + \gamma_i^{(1)}(x_i^{(1,i)}) + d^t y^{(1,i)} ,$$

$$\Gamma^{(2,i)} = \sum_{\substack{j=1\\ j\neq i}}^{n} \gamma_j(x_j^{2,i}) + \gamma_i^{(2)}(x_i^{(2,i)}) + d^t y^{(2,i)} ,$$

$$\psi^{(1,i)} = \varphi(x^{(1,i)}) + d^t y^{(1,i)}, \text{ and}$$

$$\psi^{(2,i)} = \varphi(x^{(2,i)}) + d^t y^{(2,i)},$$

then we get

THEOREM 3. *Let*

$$\Gamma = \max_{i\in N\cup B} \{\min\{\Gamma^{(1,i)}, \Gamma^{(2,i)}\}\} \text{ and } \psi = \min_{i\in N\cup B} \{\psi^{(1,i)}, \psi^{(2,i)}\} .$$

If $x^* \in R$ *then*

$$\Gamma(x') + d^t y' \leq \Gamma \leq \psi^* \leq \min\{\psi, \psi(x', y')\} .$$

Proof. The upper bound on ψ^* is obvious since for all $i \in N \cup B$, both $(x^{(1,i)}, y^{(1,i)}) \in \Omega$ and $(x^{(2,i)}, y^{(2,i)}) \in \Omega$. Thus, $\psi^* \leq \varphi(x^{(1,i)}, y^{(1,i)}$ and $\psi^* \leq \psi(x^{(2,i)}, y^{(2,i)})$ for all $i \in N \cup B$.

To obtain the lower bounds on ψ^*, let $R_i^{(1)} = \{x \in \mathbf{R}^n : x \in R \text{ and } \beta_{i_1} \leq x_i \leq x_i'\}$ and $R_i^{(2)} = \{x \in \mathbf{R}^n : x \in R \text{ and } x_i' \leq x_i \leq \beta_{i_2}\}$ for $i = 1, \ldots, n$. Since $x^* \in R$, then for any $i \in N \cup B$, either $x^* \in R_i^{(1)}$ or $x^* \in R_i^{(2)}$.

Case (1). If $x^* \in R_i^{(1)}$ then

$$\Gamma^{(1,i)} \equiv \min_{(x,y)\in\Omega\cap R'} \sum_{\substack{j=1\\ j\neq i}}^{n} \gamma_j(x_j) + \gamma_i^{(1)}(x_i) + d^t y \leq \sum_{\substack{j=1\\ j\neq i}}^{n} \gamma_j(x_j^*) + \gamma_i^{(1)}(x_i^*) + d^t y^*$$

$$\leq \sum_{j=1}^{n} \varphi_j(x_j^*) + d^t y^* = \psi^* .$$

This last inequality holds since $\gamma_j(x_j) \leq \varphi_j(x_j)$ for all $x_j \in [\beta_{j_1}, \beta_{j_2}]$ and $j =$

$1, \ldots, n$, and $\gamma_i^{(1)}(x_i) \leq \varphi_i(x_i)$ for all $x_i \in [\beta_{i_1}, x_i']$ and $x_i^* \in [\beta_{i_1}, x_i']$. Hence, $\Gamma^{(1,i)} \leq \psi^*$.

Case (2). If $x^* \notin R_i^{(1)}$, then if $x^* \in R_i^{(2)}$ (since $x^* \in R$) so that

$$\Gamma^{(2,i)} \equiv \min_{(x,y)\in\Omega\cap R'} \sum_{\substack{j=1 \\ j\neq i}}^{n} \gamma_j(x_j) + \gamma_i^{(2)}(x_i) + d^t y \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} \gamma_j(x_j^*) + \gamma_i^{(2)}(x_i^*) + d^t y^*$$

$$\leq \sum_{j=1}^{n} \varphi_j(x_j^*) + d^t y^* = \psi^* .$$

Again, this last inequality holds since $\gamma_j(x_j) \leq \varphi_j(x_j)$ for all $x_j \in [\beta_{j_1}, \beta_{j_2}]$ and $j = 1, \ldots, n$, and $\gamma_i^{(2)}(x_i) \leq \varphi_i(x_i)$ for all $x_i \in [x_i', \beta_{i_2}]$ and $x_i^* \in [x_i', \beta_{i_2}]$. Hence, $\Gamma^{(2,i)} \leq \psi^*$. Thus, for any $i \in N \cup B$, $\min\{\Gamma^{(1,i)}, \Gamma^{(2,i)}\} \leq \psi^*$, so that

$$\Gamma \equiv \max_{i\in N\cup B} \{\min\{\Gamma^{(1,i)}, \Gamma^{(2,i)}\}\} \leq \psi^* .$$

To show that $\Gamma(x') + d^t y' \leq \Gamma$, note that both $\gamma_i(x_i) \leq \gamma_i^{(1)}(x_i)$ and $\gamma_i(x_i) \leq \gamma_i^{(2)}(x_i)$ for all $x_i \in [\beta_{i_1}, \beta_{i_2}]$ and $i \in N \cup B$. Hence, for any $i \in N \cup B$,

$$\Gamma(x') + d^t y' = \min_{(x,y)\in\Omega R'} \Gamma(x) + d^t y = \min_{(x,y)\in\Omega\cap R'} \sum_{j=1}^{n} \gamma_j(x_j) + d^t y$$

$$\leq \min_{(x,y)\in\Omega\cap R'} \sum_{\substack{j=1 \\ j\neq i}}^{n} \gamma_j(x_j) + \gamma_i^{(1)}(x_i) + d^t y = \Gamma^{(1,i)} .$$

Likewise, for any $i \in N \cup B$, $\Gamma(x') + d^t y' \leq \Gamma^{(2,i)}$. Hence,

$$\Gamma(x') + d^t y' \leq \min_{i\in N\cup B} \{\min\{\Gamma^{(1,i)}, \Gamma^{(2,i)}\}\} \leq \Gamma . \qquad \square$$

As before, if $\psi - \Gamma \leq \varepsilon$ then the point $(u, v) = \text{argmin}(\psi)$ can be accepted as the global minimum vertex. If $\psi - \Gamma$ still exceeds ε, then even though (u, v) may not be the global minimum vertex, it may be possible that the solutions provided by the $2|N \cup B|$ linear programs can be used to eliminate portions of Ω that cannot contain (x^*, y^*).

More precisely, let $i \in N \cup B$. If $\Gamma^{(1,i)} \geq \psi^{(1,i)}$, then it can be shown (Phillips and Rosen 1988) that $x^* \notin R_i^{(1)}$ and hence $R_i^{(1)}$ can be discarded from consideration. Likewise, if $\Gamma^{(2,i)} \geq \psi^{(2,i)}$, then $x^* \notin R_i^{(2)}$ and hence $R_i^{(2)}$ can be discarded from consideration. Clearly, if both $\Gamma^{(1,i)} \geq \psi^{(1,i)}$ and $\Gamma^{(2,i)} \geq \psi^{(2,i)}$ then equality holds, and the point (u, v) is, in fact, the global minimum point. If some subregion is eliminated, then the hyperrectangle R can be reduced in size, and the linear underestimating program described in Theorem 1 can be repeated on this smaller region. If no subregions can be eliminated, then, as shown in Phillips and Rosen (1990), a bisection of the region R into two smaller subregions may be applied so that the initial linear underestimating program can be repeated on both of these smaller regions concurrently.

In fact, in order to prove convergence of the algorithm to an ε-approximate solution, it is necessary to require that a sufficiently large fraction of the current hyperrectangle be eliminated at every step. By defining the "length" of a side $[\beta_{s_1}, \beta_{s_2}]$ of R to be the product $\lambda_s(\beta_{s_2} - \beta_{s_1})^2$ where λ_s is some positive constant, then if the length of the longest side of R prior to the subregion elimination step is greater than some fraction δ of its current (i.e., after elimination) length (where $0 < \delta < 1$ is user specified), then not enough of R was eliminated (e.g., no subregions were eliminated and so R has not changed). Hence, eliminate whatever subregions can be eliminated (if any), *and* bisect the new region along that "longest" edge. More precisely, if a bisection of R is to be performed, then let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive constants (whose choice is specified below) and select the bisected edge/direction s such that

$$\lambda_s(\beta_{s_2} - \beta_{s_1})^2 = \max_{i=1,\ldots,n} \lambda_i(\beta_{i_2} - \beta_{i_1})^2,$$

and then set $\beta_3 := (\beta_{s_1} + B_{s_2})/2$, $R_1 := \{x : x \in R \text{ and } x_s \leqslant \beta_3\}$, and $R_2 := \{x : x \in R \text{ and } x_s \geqslant \beta_3\}$. If the second derivative of each $\varphi_i(x_i)$ exists and is continuous on $[0, \beta_i]$ for $i = 1, \ldots, n$, then the positive constants $\lambda_i \in \mathbf{R}$ can be chosen such that $|\varphi_i''(x_i)| \leqslant \lambda_i$, for all $x_i \in [0, \beta_i]$, $i = 1, \ldots, n$. Regardless of the choice of the λ_i, the initial linear underestimating program can be repeated on both of these smaller regions in parallel, and finite convergence can be guaranteed.

3. Algorithm

Based on the previous discussion and theorems, we can present a computational algorithm for obtaining a solution to problem (GP). In the following algorithm, (u, v) represents the vertex corresponding to the "best" function value ψ (the upper bound) at any point in the method, and both of these are globally accessible values. All other values are considered to be local to the enclosing procedure.

The method consists of three main parts: procedures **ComputeRectangle, SolveProblemOverRectangle,** and **Solve2nLinearPrograms.** The main program itself is called **FastSolution.** Step 5 of **SolveProblemOverRectangle** is the test of the sufficient condition since if $g_B(\tau)(z^{(\tau)}) = \min\{g_B{}^{(j)}(z^{(j)}) : j \notin B\} \geqslant 0$, then by Theorem 2, the point (x', y') solves problem (GP) and so $\psi = \psi(u, v) = \psi(x', y')$. Steps 10 through 15 of procedure **SolveProblemOverRectangle,** and in particular the call to procedure **Solve2nLinearPrograms** in Step 11, are heuristic steps designed to accelerate convergence of the algorithm by attempting to improve the incumbent vertex (x', y') and lower bound Γ', and by attempting to eliminate subregions of the hyperrectangle R which cannot contain the global optimum vertex. The constant ρ in step 15, which is required for convergence, may be set to any arbitrary positive integer. For more details concerning these steps, see Phillips and Rosen (1988). If $\varepsilon > 0$ is the desired stopping tolerance, then the algorithm can be stated as follows:

FastSolution:

1. **ComputeRectangle**(R)
2. **SolveProblemOverRectangle**(R,Γ).
3. The global solution is ψ with corresponding vertex (u,v) and lower bound Γ.

End **FastSolution.**

Procedure **ComputeRectangle**(R):

1. Compute the enclosing hyperrectangle R by solving the multiple-cost-row linear program

   $$\beta_{i_1} = \min_{(x,y)\in\Omega} x_i$$

   for each $i = 1, \ldots, n$ to get the vertices $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)}), \ldots,$ $(x^{(n)}, y^{(n)})$ and corresponding optimal function values $\beta_{l_1}, \ldots, \beta_{n_1}$. Also solve the multiple-cost-row linear program

   $$\beta_{i_2} = \max_{(x,y)\in\Omega} x_i$$

   for each $i = 1, \ldots, n$ to get the vertices $(x^{(n+1)}, y^{(n+1)})$, $(x^{(n+2)}), y^{(n+2)}), \ldots, (x^{(2n)}), y^{(2n)})$ and corresponding optimal function values $\beta_{1_2}, \ldots, \beta_{n_2}$.
2. Set $R := \{x : \beta_{i_1} \leq x_i \leq \beta_{i_2},\ i = 1, \ldots, n\}$,

   $$\psi := \min_{i=1,\ldots,2n} \{\psi(x^{(i)}, y^{(i)})\}$$

   and $(u, v) := \text{argmin}(\psi)$.
3. Return R.

End **ComputeRectangle.**

SolveProblemOverRectangle(R, Γ):

1. Given the hyperrectangle $R = \{x : \beta_{i_1} \leq x_i \leq \beta_{i_2},\ i = 1, \ldots, n\}$, construct the linear function $\Gamma(x)$ which agrees with $\varphi(x)$ at all vertices of R. That is, let

   $$\Gamma(x) = \sum_{i=1}^{n} \gamma_i(x_i)$$

   where, for $i = 1, \ldots, n$, $\gamma_i(x_i) = c_{i_0} x_i + \gamma_{i_0}$ and

   $$c_{i_0} = \frac{1}{\beta_{i_2} - \beta_{i_1}} (\varphi_i(\beta_{i_2}) - \varphi_i(\beta_{i_1})), \text{ and}$$

   $$\gamma_{i_0} = \frac{1}{\beta_{i_2} - \beta_{i_1}} (\beta_{i_2}\varphi_i(\beta_{i_1}) - \beta_{i_1}\varphi_i(\beta_{i_2})).$$

2. Let $R' = R \times \mathbf{R}^p$ and solve the linear program

   $$\min_{(x,y)\in\Omega\cap R'} \Gamma(x) + d^t y$$

to get the candidate point (x', y'). Set $\Gamma' := \Gamma(x') + d^t y'$, $\psi' := \psi(x', y')$, $\psi := \min\{\psi, \psi'\}$, and $(u, v) := \operatorname{argmin}(\psi)$. Set $I := 1$.

3. If $(\psi - \Gamma')/|\psi| \leq \varepsilon$ then set $\Gamma := \Gamma'$ and go to step (20).
4. For $i = 1, \ldots, n$, define

$$c_{i_1} = \begin{cases} c_{i_0} & \text{if } x_1' \in \{\beta_{i_1}, \beta_{i_2}\} \\ (\varphi_i(x_i') - \varphi_i(\beta_{i_1}))/(x_1' - \beta_{i_1}) & \text{if } x_1' \in (\beta_{i_1}, \beta_{i_2}) \text{ and} \end{cases}$$

$$c_{i_2} = \begin{cases} c_{i_0} & \text{if } x_1' \in \{\beta_{i_1}, \beta_{i_2}\} \\ (\varphi_i(x_i') - \varphi_i(\beta_{i_2}))/(x_1' - \beta_{i_2}) & \text{if } x_1' \in (\beta_{i_1}, \beta_{i_2}) . \end{cases}$$

Also define $c_{i_1} = c_{i_2} = c_{i_0} = d_i$ for $i = n + 1, \ldots, n + p$, and $c_{i_1} = c_{i_2} = c_{i_0} = 0$ for all other *non-original* problem variables.

5. Let B be the ordered set of indices of the basic variables at the candidate point (x', y'), let B_i represent the index of the ith basic variable in B, and let $A_B \in \mathbf{R}^{m \times m}$ be the matrix of columns from $A = [A_1 : A_2]$ which correspond to the basic variables in B. Define $w^{(j)} = A_B^{-1} a^{(j)}$, where $a^{(j)}$ is the jth column of A. For each $j \notin B$, define z such that

$$z_i^{(j)} = \begin{cases} c_{i_1} & \text{if } i = B_k \text{ and } w_k^{(j)} > 0 \\ c_{i_2} & \text{otherwise} . \end{cases}$$

In addition, for each $j \notin B$, define the function $g_B^{(j)}$ such that

$$g_B^{(j)}(z) = z_j - \sum_{i=1}^{m} w_i^{(j)} z_{B_i} ,$$

and let $\tau \notin B$ be such that

$$g_B^{(\tau)}(z^{(t)}) = \min_{j \notin B} \{ g_B^{(j)}(z^{(j)}) \} .$$

If $g_B^{(\tau)}(z^{(\tau)}) \geq 0$ then set $\Gamma := \Gamma'$ and go to step (20).

6. Let $N = \{r : r \notin B \text{ and } g_B^{(r)}(z^{(r)}) < 0\}$. Solve the multiple-cost-row linear program

$$\min_{(x,y) \in \Omega} z^{(j)t} \begin{pmatrix} x \\ y \end{pmatrix}$$

for each $j \in N$, to get the candidate points $(x^{(j)}, y^{(j)})$. Let $\psi(x^{(k)}, y^{(k)}) = \min\{\psi(x^{(j)}, y^{(j)}) : j \in N\}$. Set $\psi := \min\{\psi, \psi(x^{(k)}, y^{(k)}\}$, and $(u, v) := \operatorname{argmin}(\psi)$.

7. If $(\psi - \Gamma')/|\psi| \leq \varepsilon$ then set $\Gamma := \Gamma'$ and go to step (20).
8. If $\psi(x^{(k)}, y^{(k)}) < \psi'$, then set $\psi' := \psi(x^{(k)}, y^{(k)})$ and $(x', y') := (x^{(k)}, y^{(k)})$ and go to step (4).
9. Let $s \in \{1, \ldots, n\}$ be such that

$$\lambda_s(\beta_{s_2} - \beta_{s1})^2 = \max_{i=1,\ldots,n} \lambda_i(\beta_{i_2} - \beta_{i_1})^2 .$$

10. Set OldVolumeOfR := volume(R), and OldLengthOfSides$S := \lambda_s(\beta_{s_2} - \beta_{s_1})^2$.

11. **Solve2nLinearPrograms**(R,Γ',ψ').
12. If $(\psi - \Gamma')/|\psi| \leq \varepsilon$ then set $\Gamma := \Gamma'$ and go to step (20).
13. If $\psi' < \psi(x', y')$ then set $(x', y') := \text{argmin}(\psi')$ and go to step (4).
14. If $\lambda_s(\beta_{s_2} - \beta_{s_1})^2 < \text{OldLengthOfSideS}$ then set $I := 1$ and go to step (4).
15. If $\text{volume}(R) < \text{OldVolumeOf}R$ and $I < \rho$ then set $I := I + 1$ and go to step (4).
16. Set $\beta_3 := (\beta_{s_1} + \beta_{s_2})/2$, $R_1 := \{x : x \in R \text{ and } x_s \leq \beta_3\}$, and $R_2 := \{x : x \in R \text{ and } x_s \geq \beta_3\}$.
17. **SolveProblemOverRectangle**(R_1, Γ_1).
18. **SolveproblemOverRectangle**(R_2, Γ_2).
19. Set $\Gamma := \max\{\Gamma', \min\{\Gamma_1, \Gamma_2\}\}$.
20. Return Γ.

End **SolveProblemOverRectangle.**

Procedure **Solve2nLinearPrograms**(R,Γ',ψ'):

1. Given the hyperrectangle $R = \{x : \beta_{i_1} \leq x_i \leq \beta_{i_2},\ i = 1, \ldots, n\}$, for each $i = 1, \ldots, n$, let $\beta_{i_3} = (\beta_{i_1} + \beta_{i_2})/2$, and construct $\gamma_i^{(1)}(x_i)$, $\gamma_i^{(2)}(x_i)$, and $\gamma_i(x_i)$ such that $\gamma_i^{(1)}(x_i) = c_{i_1}x_i + \gamma_{i_1}$, $\gamma_i^{(2)}(x_i) = c_{i_2}x_i + \gamma_{i_2}$, and $\gamma_i(x_i) = c_{i_0}x_i + \gamma_{i_0}$ where

$$c_{i_1} = \frac{1}{\beta_{i_3} - \beta_{i_1}} (\varphi_i(\beta_{i_3}) - \varphi_i(\beta_{i_1})),$$

$$c_{i_2} = \frac{1}{\beta_{i_2} - \beta_{i_3}} (\varphi_i(\beta_{i_2}) - \varphi_i(\beta_{i_3})),$$

$$c_{i_0} = \frac{1}{\beta_{i_2} - \beta_{i_1}} (\varphi_i(\beta_{i_3}) - \varphi_i(\beta_{i_1})), \text{ and}$$

$$\gamma_{i_1} = \frac{1}{\beta_{i_3} - \beta_{i_1}} (\beta_{i_3}\varphi_i(\beta_{i_1}) - \beta_{i_1}\varphi_i(\beta_{i_3})),$$

$$\gamma_{i_2} = \frac{1}{\beta_{i_2} - \beta_{i_3}} (\beta_{i_2}\varphi_i(\beta_{i_3}) - \beta_{i_3}\varphi_i(\beta_{i_2})),$$

$$\gamma_{i_0} = \frac{1}{\beta_{i_2} - \beta_{i_1}} (\beta_{i_2}\varphi_i(\beta_{i_1}) - \beta_{i_1}\varphi_i(\beta_{i_2})),$$

2. Let $R' = R \times \mathbf{R}^p$. For each $i = 1, \ldots, n$, solve the pair of linear programs

$$\min_{(x,y)\in\Omega\cap R'} \sum_{\substack{j=1 \\ j\neq i}}^{n} \gamma_j(x_j) + \gamma_i^{(1)}(x_i) + d^t y \quad \text{and}$$

$$\min_{(x,y)\in\Omega\cap R'} \sum_{\substack{j=1 \\ j\neq i}}^{n} \gamma_j(x_j) + \gamma_i^{(2)}(x_i) + d^t y$$

to get the candidate points $(x^{(1,i)}, y^{(1,i)})$ and $(x^{(2,i)}, y^{(2,i)})$, respectively.

3. Denote

$$\Gamma^{(1,i)} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \gamma_j(x_j^{(1,i)}) + \gamma_i^{(1)}(x_i^{(1,i)}) + d^t y^{(1,i)},$$

$$\Gamma^{(2,i)} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \gamma_j(x_j^{(2,i)}) + \gamma_i^{(2)}(x_i^{(2,i)}) + d^t y^{(2,i)},$$

$$\psi^{(1,i)} = \varphi(x^{(1,i)}) + d^t y^{(1,i)}, \quad \text{and}$$

$$\psi^{(2,i)} = \varphi(x^{(2,i)}) + d^t y^{(2,i)}.$$

Set

$$\psi' := \min\{\psi', \min_{i=1,\ldots,n}\{\psi^{(1,i)}, \psi^{(2,i)}\}\} \quad \text{and}$$

$$\Gamma' := \max\{\Gamma', \max_{i=1,\ldots,n}\{\min\{\Gamma^{(1,i)}, \Gamma^{(2,i)}\}\}\}.$$

Set $\psi := \min\{\psi, \psi'\}$ and $(u, v) := \operatorname{argmin}(\psi)$.

4. For each $i = 1, \ldots, n$, if $\Gamma^{(1,i)} \geqslant \psi^{(1,i)}$ then set $\beta_{i_1} := \beta_{i_3}$. Likewise, if $\Gamma^{(2,i)} \geqslant \psi^{(2,i)}$ then set $\beta_{i_2} := \beta_{i_3}$.
5. Set $R := \{x : \beta_{i_1} \leqslant x_i \leqslant \beta_{i_2},\ i = 1, \ldots, n\}$.
6. Return R, Γ', and ψ'.

End **Solve2nLinearPrograms.**

4. Error Bounds

To analyze the convergence of the algorithm, denote the hyperrectangle at iteration k by $R^{(k)} = \{x : \beta_{i_1}^{(k)} \leqslant x_i \leqslant \beta_{i_2}^{(k)},\ i = 1, \ldots, n\}$. Referring to steps (11) through (14) of the algorithm, at the end of iteration k the edge s of "longest length" satisfies $\lambda_s(\beta_{s_2}^{(k+1)} - \beta_{s_1}^{(l+1)})^2 \leqslant \max\{\lambda_s(\beta_{s_2}^{(k)} - \beta_{s_1}^{(k)})^2/4, \quad \delta\lambda_s(\beta_{s_2}^{(k)} - \beta_{s_1}^{(k)})^2\}$. Furthermore, by denoting the linear underestimation error at iteration k by $e_i^{(k)}(x_i) = \varphi_i(x_i) - \gamma_i^{(k)}(x_i)$ for $i = 1, \ldots, n$, where $\gamma_i^{(k)}(x_i)$ is the linear underestimator described in step (2) of the algorithm, then it can be easily shown that for all $x_i \in [\beta_{i_1}^{(k)}, \beta_{i_2}^{(k)}]$

$$e_i^{(k)}(x_i) \leqslant \frac{1}{8}\lambda_i(\beta_{i_2}^{(k)} - \beta_{i_1}^{(k)})^2.$$

Since the sth term of the error $e_s^{(k)}(x_s)$ is the largest of the error terms at iteration k, and this term is factored by at least $\max\{1/4, \delta\}$ at iteration k, then at some iteration j all such "lengths" $\lambda_i(\beta_{i_2}^{(j)} - \beta_{i_1}^{(j)})^2$ $i = 1, \ldots, n$, will satisfy

$$e_i^{(k)}(x_i) \leqslant \frac{1}{8}\lambda_i(\beta_{i_2}^{(j)} - \beta_{i_1}^{(j)})^2 \leqslant \varepsilon/n \quad \text{for } i = 1, \ldots, n.$$

Hence, at some finite iteration j, the error between $\varphi_i(x_i)$ and $\gamma_i^{(j)}(x_i)$ over $R^{(j)}$ is bounded above by ε/n for all $i = 1, \ldots, n$, and thus the error between ψ and Γ is bounded above by ε and an ε-approximate solution has been identified.

5. Example

As an example, consider the concave quadratic function $\psi(x, y) = \varphi(x) + d^t y = -1/2(2x_1^2 + 8x_2^2)$. Note that there are no purely linear variables y in this example so that $\psi(x) \equiv \psi(x, y) \equiv \varphi(x)$ and $x \in \mathbf{R}^2$. Also define the polytope $\Omega = \{x_1, x_2) : x_1 + x_2 \leq 10,\ x_1 + 5x_2 \leq 22,\ -3x_1 + 2x_2 \leq 2,\ -x_1 - 4x_2 \leq -4,\ x_1 - 2x_2 \leq 4\}$. The vertices of Ω are $(0, 1)$, $(4, 0)$, $(8, 2)$, $(7, 3)$, and $(2, 4)$ with corresponding function values $\psi(x) = -4$, -16, -80, -85, and 68. Thus, $\psi^* = \psi(7, 3) = -85$. The feasible region Ω is shown in Figure 1.

Let $\varepsilon = 0$ and consider the solution procedure used by the algorithm. The initial hyperrectangle $R = \{x : 0 \leq x_1 \leq 8,\ 0 \leq x_2 \leq 4\}$ and the initial global incumbent function value $\psi = -80$ at the vertex $u = (8, 2)$. The initial linear underestimator of $\psi(x)$ over R is $\Gamma(x) = -8x_1 - 16x_2$ which attains its minimum over Ω at $x' = (7, 3)$ with $\Gamma(7, 3) = -104$ and $\psi(7, 3) = -85$. Hence, $x' = u = (7, 3)$, $\Gamma' = -104$, and $\psi' = \psi = -85$ (i.e., the global incumbent has been updated). Since the current incumbent vertex $x' = (7, 3)$ is strictly interior to R, then $n_0 = 2$ and the four linear functions described prior to Theorem 1 are given by $\gamma_1^{(1)}(x_1) = -7x_1$, $\gamma_1^{(2)}(x_1) = -15x_1 - 36$, $\gamma_2^{(1)}(x_2) = -12x_1$, and $\gamma_2^{(2)}(x_2) = -28x_1 - 64$. The ordered set of basic variables at the vertex $(7, 3)$ is $B = \{x_1, x_6, x_3, x_2, x_7\}$ and so only $z^{(4)}$ and $z^{(5)}$ need to be computed (since $m + n = 7$). Constructing $z^{(j)}$ as described prior to Theorem 2, $z^{(4)} = (-15, -28, 0, 0, 0, 0, 0)$ and $z^{(5)} = (-7, -28, 0, 0, 0, 0, 0)$. Hence, $g_B^{(4)}(z^{(4)}) = 0$ and $g_B^{(5)}(z^{(5)}) = 1.75$ and the sufficient conditions are satisfied.

Since the sufficient conditions have been satisfied at the vertex $(7, 3)$, the global incumbent function value $\psi = -85$ with corresponding vertex $(7, 3)$ is the global minimum; that is, $\psi^* = -85$ and $x^* = (7, 3)$. Note that the lower bound is still only $\Gamma' = -104$.

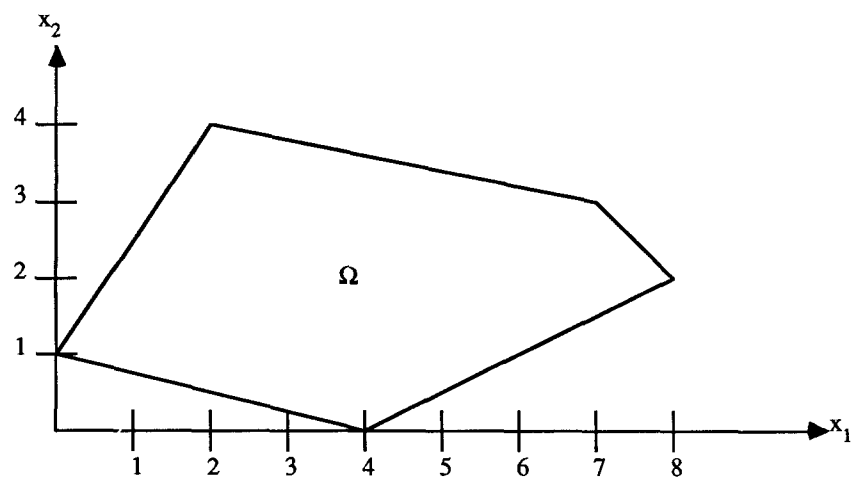

Fig. 1. The feasible region.

6. Computational Results

The computational results in this section were obtained on a Macintosh IIsi (20 MHz) and, in all of the tests cited, a stopping tolerance of $\varepsilon = 0.001$ was used. In addition, step (6) of the algorithm states that for $N = \{r : r \notin B \text{ and } g_B^{(r)}(z^{(r)}) < 0\}$ one should solve the multiple-cost-row linear program

$$\min_{(x,y)\in\Omega} z^{(j)t}\begin{pmatrix} x \\ y \end{pmatrix}$$

for each $j \in N$, to get the candidate points $(x^{(j)}, y^{(j)})$. Then let $\psi(x^{(k)}, y^{(k)}) = \min\{\psi(x^{(j)}, y^{(j)}) : j \in N\}$, and set $\psi := \min\{\psi, \psi(x^{(k)}, y^{(k)})\}$, and $(u, v) := \operatorname{argmin}(\psi)$. After considerable computational testing it has been determined that if the candidate solution (u, v) is updated by at least one of these linear programs, it is also usually updated by the one having the most negative value of the function $g_B(z)$. Hence, step (6) of the *computational* implementation solves only the single linear program

$$\min_{(x,y)\in\Omega} z^{(j)t}\begin{pmatrix} x \\ y \end{pmatrix}$$

where $g_B^{(j)}(z^{(j)}) = \min\{g_B^{(r)}(z^{(r)}) : r \in N\}$.

The class of problems tested were randomly generated concave quadratic functions of the following form:

$$\varphi(x) = \sum_{i=1}^{n} \lambda_i (x_i - v_i)^2 ,$$

where v is the unconstrained global maximum of $\varphi(x)$, and $\lambda_i < 0$ for $i = 1, \ldots, n$. It has been observed (Phillips, Rosen, and van Vliet 1991) that for problems with the global maximum $v \in \Omega$, there exist significantly more local minima than for problems of the same dimension with no restriction, on v, hence these problems should be more difficult to solve. For this reason, all problems were generated in such a way that $v \in \Omega$. Finally, five problems of each size with values of $m = 10$, $n \in \{10, 25, 50\}$, and $p \in \{0, 25, 50\}$ were tested.

A summary of the results is presented in Table I. All times reported (min, avg, and max) are CPU seconds required to solve the problem with the sufficient conditions test included. The column labeled "% Sufficient Conditions Satisfied" lists the percentage of problems solved for which the sufficient conditions were satisfied for *at least one* subproblem. The columns labeled "Time Ratio" list the ratio T_{sc}/T, where T_{sc} is the CPU time required to solve the problem using the sufficient conditions test, and T is the CPU time required to solve the problem *without* the sufficient conditions test. It is clear that the additional computation necessary to test the sufficient conditions means that the ratio $T_{sc}/T < 1$ only if the total number of iterations required is reduced enough to more than compensate for the extra computation.

Based on a simple geometric viewpoint, it would seem clear that if the linear

Table I. Concave quadratic test problems

m	n	p	CPU time (secs)			% Sufficient conditions satisfied	Time ratio		
			min	avg	max		min	avg	max
10	10	0	3.52	8.91	18.00	100	0.69	1.08	1.36
10	10	25	7.00	11.81	17.98	40	1.11	1.17	1.26
10	10	50	7.57	17.52	28.06	60	0.94	1.10	1.21
10	25	0	15.33	40.10	93.60	100	0.96	1.04	1.18
10	25	25	12.25	36.06	63.10	40	0.94	1.16	1.49
10	25	50	43.45	80.14	134.40	40	1.14	1.32	1.45
10	50	0	26.55	57.85	128.73	60	0.93	1.06	1.18
10	50	25	32.88	73.27	105.50	20	1.04	1.10	1.22
10	50	50	39.45	207.89	365.07	20	1.00	1.25	1.48

Table II. Concave quadratic test problems with $m = 10$, $n = 50$, and $p = 25$

CPU Time (secs)	
$T(\theta = 0.005)$	$T(\theta = 1)$
29.42	69.23
29.60	32.88
31.93	105.50
32.87	58.40
32.58	100.33

term $d^t y$ dominates the nonlinear function $\varphi(x)$ over Ω, then the problem is more linear in form and both the sufficient conditions and the ε-tolerance test should be satisfied more quickly. To verify this prediction, the set of concave quadratic test problems with $m = 10$, $n = 50$, and $p = 25$ were tested again but with the nonlinear term $\varphi(x)$ *dampened* by a factor θ. That is, the function minimized was $\theta\varphi(x) + d^t y$, where $\theta = 0.005$ was chosen. In addition, the ε-tolerance test in step 3 was removed so that termination could only occur if the sufficient conditions were satisfied in step 5 or if the ε-test was satisfied in steps 7 or 12. The results comparing the cases $\theta = 0.005$ and $\theta = 1$ (the original dampening factor) are given in Table II. In four of the five cases with $\theta = 0.005$, the sufficient conditions were satisfied immediately following the initial linear underestimator; hence, a total of 101 linear programs were solved. The fifth case required one improvement in the incumbent vertex (over the vertex found by the initial linear underestimator) and thus a total of 203 linear programs. The ε-tolerance was satisfied for this problem in step 12 after the $2n$ subregion linear programs were solved. On the other hand, for $\theta = 1$ four of the five problems tested required at least one domain split, and the sufficient conditions were satisfied in only one case.

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